

AN ESTIMATE ON RIEMANNIAN MANIFOLDS OF DIMENSION 4.

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ABSTRACT. We give an estimate of type $\sup \times \inf$ on Riemannian manifold of dimension 4 for a Yamabe type equation.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we deal with the following Yamabe type equation in dimension $n = 4$:

$$\Delta_g u + hu = 8u^3, \quad u > 0, \quad (1)$$

Here, Δ_g is the Laplace-Beltrami operator and h is an arbitrary bounded function..

The equation (1) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2-4], [11], [15]. In this case we have a $\sup \times \inf$ inequality. The corresponding equation in two dimensions on open set Ω of \mathbb{R}^2 , is:

$$\Delta u = V(x)e^u, \quad (2)$$

The equation (2) was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, \inf_{\Omega} u, K, \Omega).$$

And, precisely, in [8], [12], [16], and [20], we have:

$$C \sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, K, \Omega),$$

and,

$$\sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, \|V\|_{C^\alpha(\Omega)}, K, \Omega).$$

where K is a compact subset of Ω , C is a positive constant which depends on $\frac{\inf_{\Omega} V}{\sup_{\Omega} V}$, and, $\alpha \in (0, 1]$. When $6h = R_g$ the scalar curvature, and M compact, the equation (1) is Yamabe equation. T. Aubin and R. Schoen have proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary. When M is a compact Riemannian manifold, there exist some compactness result for equation (1) see [18]. Li and Zhu see [18],

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proved that the energy is bounded and if we suppose M not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose M Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product $\sup \times \inf$ is bounded. Here we extend the result of [5]. Our proof is an extension Li-Zhang result in dimension 3, see [3] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [3, 6, 11, 16, 17, 10], some applications of this method, for example an uniqueness result. We refer to [7] for the uniqueness result on the sphere and in dimension 3. Here, we give an equality of type $\sup \times \inf$ for the equation (1) in dimension 4. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3, 5]. There are other estimates of type $\sup + \inf$ on complex Monge-Ampere equation on compact manifolds, see [20-21]. They consider, on compact Kahler manifold (M, g) , the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\varphi)^n = e^{f-t\varphi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\varphi > 0 \text{ on } M \end{cases} \quad (3)$$

And, they prove some estimates of type $\sup_M + m \inf_M \leq C$ or $\sup_M + m \inf_M \geq C$ under the positivity of the first Chern class of M . Here, we have,

Theorem 1.1. *For all compact set K of M , there is a positive constant c , which depends only on, $h_0 = \|h\|_{L^\infty(M)}$, K , M , g such that:*

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all u solution of (1).

This theorem extend to the dimension 4 a result of the author and of Li and Zhang result, see [17]. Here, we use a different method than the method of Li and Zhang in [17]. Also, we extend a result of [5].

Corollary 1.2. *For all compact set K of M there is a positive constant c , such that:*

$$\sup_K u \leq c = c(g, m, h_0, K, M) \text{ if } \inf_M u \geq m > 0,$$

for all u solution of (1).

2. PROOF OF THE RESULTS

Proof of theorem 1.1:

Let x_0 be a point of M . We want to prove a uniform estimate around x_0 .

Let $(u_i)_i$ be a sequence of solutions to:

$$\Delta u_i + h u_i = 8 u_i^3, \quad u_i > 0,$$

We argue by contradiction, we assume that the $\sup \times \inf$ is not bounded.

$\forall c, R > 0 \exists u_{c,R}$ solution to (1) such that:

$$R^2 \left(\sup_{B(x_0, R)} u_{c,R} \right)^{1/3} \times \inf_{B(x_0, 2R)} u_{c,R} \geq c, \quad (4)$$

Proposition 2.1. (*blow-up analysis*)

There is a sequence of points $(y_i)_i$, $y_i \rightarrow x_0$ and two sequences of positive real numbers $(l_i)_i, (L_i)_i$, $l_i \rightarrow 0$, $L_i \rightarrow +\infty$, such that if we set $v_i(y) = \frac{u_i[\exp_{y_i}(y/[u_i(y_i)])]}{u_i(y_i)}$, we have:

$$0 < v_i(y) \leq \beta_i \leq 2, \quad \beta_i \rightarrow 1.$$

$$v_i(y) \rightarrow \frac{1}{1 + |y|^2}, \quad \text{uniformly on compact sets of } \mathbb{R}^4.$$

$$l_i^2 (u_i(y_i))^{1/3} \min_M u_i \rightarrow +\infty.$$

Proof:

We use the hypothesis (4), we take two sequences, $R_i > 0$, $R_i \rightarrow 0$ and $c_i \rightarrow +\infty$, such that,

$$R_i^2 \left(\sup_{B(x_0, R_i)} u_i \right)^{1/3} \times \inf_{B(x_0, 2R_i)} u_i \geq c_i \rightarrow +\infty, \quad (5)$$

Let, $x_i \in B(x_0, R_i)$, such that $\sup_{B(x_0, R_i)} u_i = u_i(x_i)$ and $s_i(x) = [R_i - d(x, x_i)]u_i(x)$, $x \in B(x_i, R_i)$. Then, $x_i \rightarrow x_0$.

We have:

$$\max_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i u_i(x_i) \geq \sqrt{c_i} \rightarrow +\infty.$$

We set :

$$l_i = R_i - d(y_i, x_i), \quad \bar{u}_i(y) = u_i[\exp_{y_i}(y)], \quad v_i(z) = \frac{u_i[\exp_{y_i}(z/[u_i(y_i)])]}{u_i(y_i)}.$$

Clearly, we have, $y_i \rightarrow x_0$. We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/4}}[u_i(y_i)] = \frac{[s_i(y_i)]}{c_i^{1/4}} \geq \frac{c_i^{1/2}}{c_i^{1/4}} = c_i^{1/4} \rightarrow +\infty.$$

If $|z| \leq L_i$, then $y = \exp_{y_i}[z/[u_i(y_i)]] \in B(y_i, \delta_i l_i)$ with $\delta_i = \frac{1}{(c_i)^{1/4}}$ and $d(y, y_i) < R_i - d(y_i, x_i)$, thus, $d(y, x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$. We can write,

$$u_i(y)[R_i - d(y, y_i)] \leq u_i(y_i)l_i.$$

But, $d(y, y_i) \leq \delta_i l_i$, $R_i > l_i$ and $R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$, hence, we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \frac{l_i}{l_i(1 - \delta_i)} \leq 2.$$

We set, $\beta_i = \frac{1}{1 - \delta_i}$, clearly $\beta_i \rightarrow 1$.

The function v_i satisfies the following equation:

$$-g^{jk}(z)\partial_{jk}v_i - \partial_k \left[g^{jk}\sqrt{|g|} \right] (z)\partial_j v_i + \frac{h(z)}{[u_i(y_i)]^2}v_i = 8v_i^3 \quad (6)$$

We use Ascoli and Ladyzenskaya theorems to obtain the local uniform convergence (on every compact set of \mathbb{R}^4) of $(v_i)_i$ to v solution on \mathbb{R}^4 to:

$$\Delta v = 8v^3, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2,$$

By the maximum principle, we have $v > 0$ on \mathbb{R}^n . According to Caffarelli-Gidas-Spruck result (see [10]), we have, $v(y) = \frac{1}{1 + |y|^2}$.

Polar Geodesic Coordinates

Let u be a function on M . We set $\bar{u}(r, \theta) = u[\exp_x(r\theta)]$. We denote $g_{x,ij}$ the local expression of the metric g in the exponential chart centered at x .

We set,

$$w_i(t, \theta) = e^t \bar{u}_i(e^t, \theta) = e^t u_i[\exp_{y_i}(e^t \theta)],$$

$$a(y_i, t, \theta) = \log J(y_i, e^t, \theta) = \log[\sqrt{\det(g_{y_i,ij})}].$$

We can write the Laplace-Beltrami operator in polar geodesic coordinates:

$$-\Delta u = \partial_{rr} \bar{u} + \frac{3}{r} \partial_r \bar{u} + \partial_r [\log J(x, r, \theta)] \partial_r \bar{u} - \frac{1}{r^2} \Delta_\theta \bar{u}. \quad (7)$$

We deduce the two following lemmas:

Lemma 2.2. *The function w_i is a solution to:*

$$-\partial_{tt} w_i - \partial_t a \partial_t w_i - \Delta_\theta w_i + c w_i = 8 w_i^3, \quad (8)$$

with

$$c = c(y_i, t, \theta) = 1 + \partial_t a + h e^{2t},$$

Proof:

We write:

$$\partial_t w_i = e^{2t} \partial_r \bar{u}_i + w_i, \quad \partial_{tt} w_i = e^{3t} \left[\partial_{rr} \bar{u}_i + \frac{3}{e^t} \partial_r \bar{u}_i \right] + w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \quad \partial_t a \partial_t w_i = e^{3t} [\partial_r \log J \partial_r \bar{u}_i] + \partial_t a w_i.$$

Lemma 1 follows.

Let $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$. We can write:

$$-\frac{1}{\sqrt{b_1}}\partial_{tt}(\sqrt{b_1}w_i) - \Delta_\theta w_i + [c(t) + b_1^{-1/2}b_2(t, \theta)]w_i = 8w_i^3,$$

$$\text{where, } b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}}\partial_{tt}b_1 - \frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2.$$

We set,

$$\tilde{w}_i = \sqrt{b_1}w_i.$$

Lemma 2.3. *The function \tilde{w}_i is a solution to:*

$$\begin{aligned} -\partial_{tt}\tilde{w}_i + \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)\tilde{w}_i = \\ = 8 \left(\frac{1}{b_1} \right) \tilde{w}_i^3, \end{aligned} \tag{9}$$

where, c_2 is a function to be determined.

Proof:

We have:

$$-\partial_{tt}\tilde{w}_i - \sqrt{b_1}\Delta_\theta w_i + (c + b_2)\tilde{w}_i = 8 \left(\frac{1}{b_1} \right) \tilde{w}_i^3,$$

But,

$$\Delta_\theta(\sqrt{b_1}w_i) = \sqrt{b_1}\Delta_\theta w_i - 2\nabla_\theta w_i \cdot \nabla_\theta \sqrt{b_1} + w_i\Delta_\theta(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1}w_i) = w_i\nabla_\theta\sqrt{b_1} + \sqrt{b_1}\nabla_\theta w_i,$$

we can write,

$$\nabla_\theta w_i \cdot \nabla_\theta \sqrt{b_1} = \nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) - \tilde{w}_i |\nabla_\theta \log(\sqrt{b_1})|^2,$$

we deduce,

$$\sqrt{b_1}\Delta_\theta w_i = \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) - c_2 \tilde{w}_i,$$

with $c_2 = [\frac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$. Lemma 2 is proved.

The moving-Plane method:

Let ξ_i be a real number, we assume $\xi_i \leq t$. We set $t^{\xi_i} = 2\xi_i - t$ and $\tilde{w}_i^{\xi_i}(t, \theta) = \tilde{w}_i(t^{\xi_i}, \theta)$. Set, $\lambda_i = -\log u_i(y_i)$

Proposition 2.4. *We claim:*

$$\tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) \geq \tilde{k} > 0, \quad \forall \theta \in \mathbb{S}_3. \quad (10)$$

For all $\beta > 0$, there exists $c_\beta > 0$ such that:

$$\frac{1}{c_\beta} e^t \leq \tilde{w}_i(\lambda_i + t, \theta) \leq c_\beta e^t, \quad \forall t \leq \beta, \quad \forall \theta \in \mathbb{S}_3. \quad (11)$$

Proof:

As in [2], we have, $w_i(\lambda_i, \theta) - w_i(\lambda_i + 4, \theta) \geq k > 0$ for i large, $\forall \theta$. We can remark that $b_1(y_i, \lambda_i, \theta) \rightarrow 1$ and $b_1(y_i, \lambda_i + 4, \theta) \rightarrow 1$ uniformly in θ , we obtain the first claim of proposition 2.4. For the second claim we use proposition 2.1, see also [2].

We set:

$$\bar{Z}_i = -\partial_{tt}(\dots) + \Delta_\theta(\dots) + 2\nabla_\theta(\dots) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)(\dots) \quad (12)$$

Remark : In the operator \bar{Z}_i , we can remark that:

$$c + b_1^{-1/2}b_2 - c_2 \geq k' > 0, \quad \text{for } t \ll 0,$$

we can apply the maximum principle and the Hopf lemma.

Goal:

Like in [2], we have an elliptic second order operator. Here it is \bar{Z}_i , the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq 0, \text{ if } \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq 0. \quad (13)$$

We write, $\Delta_\theta = \Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}$. We obtain:

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &= (\Delta_{g_{y_i, e^{t\xi_i}, \mathbb{S}_3}} - \Delta_{g_{y_i, e^t, \mathbb{S}_3}})(\tilde{w}_i^{\xi_i}) + \\ &+ 2(\nabla_{\theta, e^{t\xi_i}} - \nabla_{\theta, e^t})(w_i^{\xi_i}) \cdot \nabla_{\theta, e^{t\xi_i}} \log(\sqrt{b_1^{\xi_i}}) + 2\nabla_{\theta, e^t}(\tilde{w}_i^{\xi_i}) \cdot \nabla_{\theta, e^{t\xi_i}} [\log(\sqrt{b_1^{\xi_i}}) - \log \sqrt{b_1}] + \\ &+ 2\nabla_{\theta, e^t} w_i^{\xi_i} \cdot (\nabla_{\theta, e^{t\xi_i}} - \nabla_{\theta, e^t}) \log \sqrt{b_1} - [(c + b_1^{-1/2} b_2 - c_2)^{\xi_i} - (c + b_1^{-1/2} b_2 - c_2)] \tilde{w}_i^{\xi_i} + \\ &+ 8 \left(\frac{1}{b_1^{\xi_i}} \right) (\tilde{w}_i^{\xi_i})^3 - 8 \left(\frac{1}{b_1} \right) \tilde{w}_i^3. \end{aligned} \quad (14)$$

Clearly, we have the following lemma:

Lemma 2.5.

$$b_1(y_i, t, \theta) = 1 - \frac{1}{3} Ricci_{y_i}(\theta, \theta) e^{2t} + \dots,$$

$$R_g(e^t \theta) = R_g(y_i) + \langle \nabla R_g(y_i) | \theta \rangle e^t + \dots$$

According to proposition 1 and lemma 3,

Proposition 2.6.

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &\leq 8(b_1^{\xi_i})[(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3] + C|e^{2t} - e^{2t\xi_i}|(|\nabla_\theta \tilde{w}_i^{\xi_i}| + |\nabla_\theta^2(\tilde{w}_i^{\xi_i})|) + \\ &+ C|e^{2t} - e^{2t\xi_i}|(|Ricci_{y_i}| + |h|)\tilde{w}_i^{\xi_i} + C'w_i^{\xi_i}|e^{3t\xi_i} - e^{3t}|. \end{aligned} \quad (15)$$

Proof of proposition 2.6:

In polar geodesic coordinates (and the Gauss lemma):

$$g = dt^2 + r^2 \tilde{g}_{ij}^k d\theta^i d\theta^j \text{ et } \sqrt{|\tilde{g}^k|} = \alpha^k(\theta) \sqrt{[\det(g_{x,ij})]}, \quad (16)$$

where α^k is the volume element of the unit sphere associated to U^k .

We can write (with lemma 2.3):

$$|\partial_t b_1(t)| + |\partial_{tt} b_1(t)| + |\partial_{tt} a(t)| \leq C e^{2t},$$

and,

$$|\partial_{\theta_j} b_1| + |\partial_{\theta_j, \theta_k} b_1| + |\partial_{t, \theta_j} b_1| + |\partial_{t, \theta_j, \theta_k} b_1| \leq C e^{2t},$$

But,

$$\Delta_\theta = \Delta_{g_{y_i, e^t, \mathbb{S}_3}} = - \frac{\partial_{\theta^l} [\tilde{g}^{\theta^l \theta^j}(e^t, \theta) \sqrt{|\tilde{g}^k(e^t, \theta)|} \partial_{\theta^j}]}{\sqrt{|\tilde{g}^k(e^t, \theta)|}}.$$

Then,

$$A_i := \left[\left[\frac{\partial_{\theta^l} (\tilde{g}^{\theta^l \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j})}{\sqrt{|\tilde{g}^k|}} \right]^{\xi_i} - \left[\frac{\partial_{\theta^l} (\tilde{g}^{\theta^l \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j})}{\sqrt{|\tilde{g}^k|}} \right] \right] (\tilde{w}_i^{\xi_i}) = B_i + D_i \quad (17)$$

where,

$$B_i = \left[\tilde{g}^{\theta^l \theta^j}(e^{t^{\xi_i}}, \theta) - \tilde{g}^{\theta^l \theta^j}(e^t, \theta) \right] \partial_{\theta^l \theta^j} \tilde{w}_i^{\xi_i}, \quad (18)$$

and,

$$D_i = \left[\frac{\partial_{\theta^l} [\tilde{g}^{\theta^l \theta^j}(e^{t^{\xi_i}}, \theta) \sqrt{|\tilde{g}^k|}(e^{t^{\xi_i}}, \theta)]}{\sqrt{|\tilde{g}^k|}(e^{t^{\xi_i}}, \theta)} - \frac{\partial_{\theta^l} [\tilde{g}^{\theta^l \theta^j}(e^t, \theta) \sqrt{|\tilde{g}^k|}(e^t, \theta)]}{\sqrt{|\tilde{g}^k|}(e^t, \theta)} \right] \partial_{\theta^j} \tilde{w}_i^{\xi_i}. \quad (19)$$

Clearly, we can choose $\epsilon_1 > 0$ such that:

$$|\partial_r \tilde{g}_{ij}^k(x, r, \theta)| + |\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, r, \theta)| \leq Cr, \quad x \in B(x_0, \epsilon_1) \quad r \in [0, \epsilon_1], \quad \theta \in U^k. \quad (20)$$

finally,

$$A_i \leq C_k |e^{2t} - e^{2t_i}| \left[|\nabla_\theta \tilde{w}_i^{\xi_i}| + |\nabla_\theta^2(\tilde{w}_i^{\xi_i})| \right], \quad (21)$$

We take, $C = \max\{C_i, 1 \leq i \leq q\}$ and we use (14). Proposition 2.6 is proved.

We have,

$$c(y_i, t, \theta) = 1 + \partial_t a + h e^{2t}, \quad (22)$$

$$b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2, \quad (23)$$

$$c_2 = \left[\frac{1}{\sqrt{b_1}} \Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2 \right], \quad (24)$$

We assume that $\lambda \leq \lambda_i + 2 = -\log u_i(y_i) + 2$, which will be chosen later.

We work on $[\lambda, t_i] \times \mathbb{S}_3$ with $t_i = \log l_i \rightarrow -\infty$, l_i as in the proposition 1. For i large $\log l_i \gg \lambda_i + 2$.

The functions v_i tend to a radially symmetric function, then, $\partial_{\theta_j} w_i^\lambda \rightarrow 0$ if $i \rightarrow +\infty$ and,

$$\frac{\partial_{\theta_j} w_i^\lambda(t, \theta)}{w_i^\lambda} = \frac{e^{(n-2)[(\lambda-\lambda_i)+(\xi_i-t)]/2} e^{[(\lambda-\lambda_i)+(\xi_i-t)]} (\partial_{\theta_j} v_i)(e^{[(\lambda-\lambda_i)+(\lambda-t)]} \theta)}{e^{(n-2)[(\lambda-\lambda_i)+(\lambda-t)]/2} v_i[e^{(\lambda-\lambda_i)+(\lambda-t)} \theta]} \leq \bar{C}_i,$$

where \bar{C}_i does not depend on λ and tends to 0. We have also,

$$|\partial_\theta w_i^\lambda(t, \theta)| + |\partial_{\theta, \theta} w_i^\lambda(t, \theta)| \leq \tilde{C}_i w_i^\lambda(t, \theta), \quad \tilde{C}_i \rightarrow 0. \quad (25)$$

and,

$$|\partial_\theta \tilde{w}_i^\lambda(t, \theta)| + |\partial_{\theta, \theta} \tilde{w}_i^\lambda(t, \theta)| \leq \tilde{C}_i \tilde{w}_i^\lambda(t, \theta), \quad \tilde{C}_i \rightarrow 0. \quad (26)$$

\tilde{C}_i does not depend on λ .

Now, we set:

$$\bar{w}_i = \tilde{w}_i - \frac{\tilde{m}_i}{2} e^{2t}, \quad (27)$$

with, $m_i = \frac{1}{2}u_i(x_i)^{1/3} \min_M u_i$. As in [2], we have,

Lemma 2.7. *There is $\nu < 0$ such that for $\lambda \leq \nu$:*

$$\bar{w}_i^\lambda(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \quad \forall (t, \theta) \in [\lambda, t_i] \times \mathbb{S}_3. \quad (28)$$

Let ξ_i be the following real number,

$$\xi_i = \sup\{\lambda \leq \lambda_i + 2, \bar{w}_i^{\xi_i}(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \quad \forall (t, \theta) \in [\xi_i, t_i] \times \mathbb{S}_3\}.$$

Like in [2], we use the previous lemma to show:

$$\bar{w}_i^{\xi_i} - \bar{w}_i \leq 0 \Rightarrow \bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 0.$$

We have,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \tilde{w}_i) \leq 8b_1^{\xi_i}[(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3] + O(1)(e^{2t} - e^{2t^{\xi_i}}) + O(1)\tilde{w}_i^{\xi_i}(e^{2t} - e^{2t^{\xi_i}}).$$

$$-\bar{Z}_i(e^{2t^{\xi_i}} - e^{2t}) = (4 - 1 - \partial_t a - h e^{2t} + b_1^{-1/2} b_2 - c_2)(e^{2t^{\xi_i}} - e^{2t}) \leq c_3(e^{2t^{\xi_i}} - e^{2t})$$

Thus,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 8b_1^{\xi_i}[(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3] + (c_3 m_i - c_4)(e^{2t^{\xi_i}} - e^{2t}).$$

with, $c_3, c_4 > 0$.

But,

$$0 < \tilde{w}_i^{\xi_i} \leq 2e, \quad \tilde{w}_i \geq \frac{m_i}{2}e^{2t} \quad \text{and} \quad \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq \frac{m_i}{2}(e^{2t^{\xi_i}} - e^{2t}),$$

and,

$$(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3 = (\tilde{w}_i^{\xi_i} - \tilde{w}_i)[(\tilde{w}_i^{\xi_i})^2 + \tilde{w}_i^{\xi_i} \tilde{w}_i + \tilde{w}_i^2] \leq (\tilde{w}_i^{\xi_i} - \tilde{w}_i)(\tilde{w}_i^{\xi_i})^2 + (\tilde{w}_i^{\xi_i} - \tilde{w}_i) \frac{m_i^2 e^{2t}}{4} + (\tilde{w}_i^{\xi_i} - \tilde{w}_i) \frac{m_i}{2} e^t \tilde{w}_i^{\xi_i}, \quad (29)$$

then,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq [\frac{am_i^3}{16} - O(1)] + [\frac{am_i^2}{8} - O(1)]e^t \bar{w}_i^{\xi_i} (e^{2t\xi_i} - e^{2t}) \leq 0. \quad (30)$$

If we use the maximum principle and the Hopf lemma, we obtain (as in [2]):

$$\max_{\theta \in \mathbb{S}_3} w_i(t_i, \theta) \leq \min_{\theta \in \mathbb{S}_3} w_i(2\xi_i - t_i, \theta),$$

we can write (using proposition 2):

$$l_i[u_i(y_i)]^{1/3} \min_M u_i \leq c, \quad (31)$$

3. METRIC AND LAPLACIAN

In this section, we give some remarks on Polar Geodesic coordinates and the Laplacian in these coordinates. First by using the Jacobi Fields we can have an expansion of the metric in geodesic coordinates, we can extend this result to polar geodesic coordinates.

Estimate of the metric in Polar Coordinates.

Let us consider a riemannian manifold (not necessarily compact) (M, g) . We set $g_{x,ij}$ the components of the metric in the exponential chart centered at x .

By the Gauss formula we have:

$$g = ds^2 = dt^2 + g_{ij}^k(r, \theta) d\theta^i d\theta^j = dt^2 + r^2 \tilde{g}_{ij}^k(r, \theta) d\theta^i d\theta^j = g_{x,ij} dx^i dx^j,$$

in polar chart centered at x , $]0, \epsilon_0[\times U^k$, with (U^k, ψ) a chart of the unit sphere \mathbb{S}_{n-1} .

We can write the volume element as:

$$dV_g = r^{n-1} \sqrt{|\tilde{g}^k|} dr d\theta^1 \dots d\theta^{n-1} = \sqrt{[\det(g_{x,ij})]} dx^1 \dots dx^n,$$

thus,

$$dV_g = r^{n-1} \sqrt{[\det(g_{x,ij})]} [\exp_x(r\theta)] \alpha^k(\theta) dr d\theta^1 \dots d\theta^{n-1},$$

with, α^k is such that $d\sigma_{\mathbb{S}_{n-1}} = \alpha^k(\theta)d\theta^1 \dots d\theta^{n-1}$. (volume element of the sphere in the chart (U^k, ψ)).

Thus,

$$\sqrt{|\tilde{g}^k|} = \alpha^k(\theta)\sqrt{[\det(g_{x,ij})]},$$

Proposition: Let us consider $x_0 \in M$, there is $\epsilon_1 > 0$ and U^k , such that:

$$|\partial_r \tilde{g}_{ij}^k(x, r, \theta)| + |\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, r, \theta)| \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

and,

$$|\partial_r |\tilde{g}^k|(x, r, \theta)| + \partial_r \partial_{\theta^m} |\tilde{g}^k|(x, r, \theta) \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

and,

$$|\partial_r \partial_{\theta^m} \left[\frac{\sqrt{|\tilde{g}^k|}}{\alpha^k(\theta)} \right] (x, r, \theta)| + |\partial_r \partial_{\theta^m} \partial_{\theta^{m'}} \left[\frac{\sqrt{|\tilde{g}^k|}}{\alpha^k(\theta)} \right] (x, r, \theta)| \leq Cr \quad \forall (x, r, \theta) \in B(x_0, \epsilon_1) \times [0, \epsilon_1] \times U^k.$$

Proof:

Next, we use Einstein convention:

First, we consider a chart (Ω, φ) in x_0 , such that $\bar{\Omega}$ is compact, we can assume it normal at x_0 .

According to lemma 2.3.7 of [He], for all $x_0 \in M$ there is $\epsilon_0 > 0$ such that the application $u : (x, v) \rightarrow \exp_x(v)$ on $B(x_0, \epsilon_0) \times B(0, \epsilon_0)$ in M is C^∞ and for all $x \in B(x_0, \epsilon)$ the application $v \rightarrow \exp_x(v)$ is a diifeomorphism of $B(0, \epsilon_0)$ in $B(x_0, \epsilon_0)$ with $\exp_x[\partial B(0, \mu)] = \partial B(x, \mu)$, $\mu \leq \epsilon_0$. Without loss of generality we can assume that $B(x, \epsilon_0) \subset \subset \Omega$ for all $x \in B(x_0, \epsilon_0)$.

Thus, we have for all $x \in B(x_0, \epsilon_0)$, $[B(x, \epsilon_0), \exp_x^{-1}]$ is a normal chart in x . (In this case we can define polar coordinates). For all $x \in B(x_0, \epsilon_0)$:

$$g_{x,ij}(z) = g(z)(\partial_{z^i,x}, \partial_{z^j,x}),$$

with $\partial_{z^i,x}$ is the canonical vector field in the exponential chart.

We set $a_i^k(z, x) = \frac{(\varphi \circ \exp_x)^k}{\partial z^i}[\exp_x^{-1}(z)]$, then $\partial_{z^i, x} = a_i^k(z, x)\partial_{u^k, \varphi}$,

with $\partial_{u^k, \varphi}$, the canonical vector field with respect to chart (Ω, φ) , this vector field do not depends on x and the functions a_i^k are regular of z and x . We obtain,

$$g_{x, ij}(z) = g(z)[a_i^k(z, x)\partial_{u^k, \varphi}; a_j^l(z, x)\partial_{u^l, \varphi}] = a_i^k(z, x)a_j^l(z, x)g_{kl}(z),$$

with g_{kl} the component of g in the chart (Ω, φ) .

We have $z = \exp_x(y)$, $y \in B(0, \epsilon_0) \subset \mathbb{R}^n$ and $y = r\theta$ in polar coordinates, thus $(x, r, \theta) \rightarrow g_{x, ij}[\exp_x(r\theta)]$ is C^∞ of x, r et θ .

We have, by definition, $g_{ij}^k(r\theta) = g_{[\exp_x(r\theta)]}(\partial_{\theta^i, x}, \partial_{\theta^j, x})$, (canonical vector fields).

We can write,

$$\partial_{\theta^i, x} = r b_i^k(\theta)\partial_{z^k, x},$$

with b_i^j regular. (Note, that here, we can use the function θ^i , as regular function in the chart of the sphere. With this procedure we can deduce the component of the metric in polar coordinates (a good expansion)). Thus,

$$g_{ij}^k(r, \theta) = r^2 g[\exp_x(r\theta)](b_i^k\partial_{z^k, x}, b_j^l\partial_{z^l, x}),$$

Then,

$$g_{ij}^k(r, \theta) = r^2 b_i^k(\theta)b_j^l(\theta)g_{x, kl}[\exp_x(r\theta)].$$

Thus, the functions $\tilde{g}_{ij}^k : (x, r, \theta) \rightarrow b_j^k b_j^l g_{x, kl}[\exp_x(r\theta)]$ is regular of x, r and θ . We have,

$$\partial_r \tilde{g}_{ij}^k(x, 0, \theta) = b_i^k(\theta)b_j^l(\theta)c^m(\theta)\partial_m g_{x, kl}(x) = 0,$$

because the exponential chart is normal in x and $g_{x, kl}$ are the component of g in this chart. We have:

$$\partial_{\theta^m} \tilde{g}_{ij}^k(x, r, \theta) = \tilde{b}_i^k(\theta)\tilde{b}_j^l(\theta)g_{x, kl}[\exp_x(r\theta)] + r\bar{b}_i^k(\theta)\bar{b}_j^l(\theta)\bar{c}_m^s(\theta)\partial_s g_{x, kl}[\exp_x(r\theta)],$$

We also have:

$$\begin{aligned}\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, r, \theta) &= u_{ijmr}^{klq}(\theta) \partial_q g_{x,kl}[\exp_x(r\theta)] + v_{ijmr}^{kl}(\theta) w^s(t) \partial_s g_{x,kl}[\exp_x(r\theta)] + \\ &+ r h_{ijrm}^{klst}(\theta) \partial_{st} g_{x,kl}[\exp_x(r\theta)].\end{aligned}$$

Thus,

$$\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, 0, \theta) = 0, \quad \forall x \in B(x_0, \epsilon_0), \quad \forall \theta \in U^k.$$

Thus, we obtain:

$$\partial_r \tilde{g}_{ij}^k(x, 0, \theta) = \partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, 0, \theta) = 0, \quad \forall x \in B(x_0, \epsilon_0), \quad \forall \theta \in U^k. \quad (*)$$

Because $\sqrt{|\tilde{g}^k|} = \alpha^k(\theta) \sqrt{[\det(g_{x,ij})]}$, we deduce,

$$\partial_r (\log \sqrt{|\tilde{g}^k|}) = \partial_r [\log(\sqrt{[\det(g_{x,ij})])}]$$

We use the definition of the determinant

$$\det[g_{x,ij}][\exp_x(r\theta)] = \Sigma \Pi g_{x,kl}[\exp_x(r\theta)],$$

Thus,

$$\partial_r \det[g_{x,ij}](x) = \Sigma \Pi [g_{x,kl}(x)] a^s(\theta) \partial_s g_{x,mn}(x) = 0,$$

because the exponential chart is normal at x .

Finally

$$\partial_r |\tilde{g}^k|(x, 0, \theta) = 0, \quad \forall x \in B(x_0, \epsilon_0), \quad \forall \theta \in U^k.$$

We also have $\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, 0, \theta)$, to prove that,

$$\partial_r \partial_{\theta^m} |\tilde{g}^k|(x, 0, \theta) = 0.$$

If we set $D_m = \partial_{\theta^m} \frac{\sqrt{|\tilde{g}^k|}}{\alpha^k(\theta)}(x, r, \theta)$,

$$D_m = r \Sigma \Pi \beta_m^l(\theta) \partial_l g_{x,ij} [\exp_x(r\theta)] g_{x,ij} [\exp_x(r\theta)],$$

We have $D_m(x, 0, \theta) = 0$ and,

$$\partial_r D_m(x, 0, \theta) = \lim_{r \rightarrow 0} (D_m/r)(x, r, \theta) = 0,$$

Thus

$$\partial_r \partial_{\theta_m} \left[\frac{\sqrt{|\tilde{g}^k|}}{\alpha^k(\theta)} \right] (x, 0, \theta) = 0,$$

$$\partial_{\theta_{m'}} D_m = r \Sigma \Pi \partial_{m'} \beta_m^l \partial_l g_{x,ij} g_{x,ij} + r^2 \Sigma \Pi \beta_m^l \beta_{m'}^{l'} \partial_{l'} g_{x,ij} g_{x,ij} + r^2 \Sigma \Pi \beta_m^l \beta_{m'}^{l'} \partial_l g_{x,ij} \partial_{l'} g_{x,ij} g_{x,ij},$$

but, $\partial_{\theta_{m'}} D_m(x, 0, \theta) = 0$, we have,

$$\partial_r \partial_{\theta_{m'}} D_m(x, 0, \theta) = \lim_{r \rightarrow 0} [\partial_{\theta_{m'}} D_m/r] = 0,$$

Thus,

$$\partial_r \partial_{\theta_m} \partial_{\theta_{m'}} \left[\frac{\sqrt{|\tilde{g}^k|}}{\alpha^k(\theta)} \right] (x, 0, \theta) = 0,$$

Finally

$$\partial_r \tilde{g}_{ij}^k(x, 0, \theta) = \partial_r \partial_{\theta_m} \tilde{g}_{ij}^k(x, 0, \theta) = 0 \quad \forall x \in B(x_0, \epsilon_0), \quad \forall \theta \in U^k. \quad (**)$$

$$\partial_r |\tilde{g}^k|(x, 0, \theta) = \partial_r \partial_{\theta_m} |\tilde{g}^k|(x, 0, \theta) = 0 \quad \forall x \in B(x_0, \epsilon_0), \quad \forall \theta \in U^k. \quad (***)$$

$$\partial_r \partial_{\theta_m} \left[\frac{\sqrt{|\tilde{g}^k|}}{\alpha^k(\theta)} \right] (x, 0, \theta) = \partial_r \partial_{\theta_m} \partial_{\theta_{m'}} \left[\frac{\sqrt{|\tilde{g}^k|}}{\alpha^k(\theta)} \right] (x, 0, \theta) = 0 \quad \forall x \in B(x_0, \epsilon_0), \quad \forall \theta \in U^k. \quad (****)$$

We can reduce the open set U^k to have these estimates and use the uniform continuity to have the estimates.

The Laplacian in polar coordinates

We can write in $[0, \epsilon_1] \times U^k$,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log \sqrt{|\tilde{g}^k|}] \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

On a,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r \log J(x, r, \theta) \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We can write the Laplacian (radial and angular decomposition, see S. Lang book),

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log J(x, r, \theta)] \partial_r - \Delta_{\mathbb{S}_r(x)},$$

with $\Delta_{\mathbb{S}_r(x)}$ the Laplacian on $\mathbb{S}_r(x)$.

Locally we can write $\Delta_{\mathbb{S}_r(x)}$ as:

$$-\Delta_{\mathbb{S}_r(x)} = \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We set: $L_\theta(x, r)(\dots) = r^2 \Delta_{\mathbb{S}_r(x)}(\dots)[\exp_x(r\theta)]$.

The operator $L_\theta(x, r)$ act on $C^2(\mathbb{S}_{n-1})$ functions globally and not depends on the chart on \mathbb{S}_{n-1} and locally we can write it as:

$$L_\theta(x, r) = -\frac{1}{\sqrt{|\tilde{g}^k|}} \partial_{\theta^i} [\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}].$$

We have

$$\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [J(x, r, \theta)] \partial_r - \frac{1}{r^2} L_\theta(x, r).$$

We set, for u a function on M , $\bar{u}(r, \theta) = u \circ \exp_x(r\theta)$ in polar coordinates centered in x :

$$-\Delta u = \partial_{rr}\bar{u} + \frac{n-1}{r}\partial_r\bar{u} + \partial_r[J(x, r, \theta)]\partial_r\bar{u} - \Delta_{\mathbb{S}_r(x)}(u|_{\mathbb{S}_r(x)})[\exp_x(r\theta)],$$

$$r^2\Delta_{\mathbb{S}_r(x)}(u|_{\mathbb{S}_r(x)})[\exp_x(r\theta)] = -\frac{1}{\sqrt{|\tilde{g}^k|}}\partial_{\theta^i}\left[\tilde{g}^{i\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j}[u\circ\exp_x(r\theta)]\right] = L_\theta(x, r)\bar{u}.$$

Thus,

$$-\Delta u = \partial_{rr}\bar{u} + \frac{n-1}{r}\partial_r\bar{u} + \partial_r[J(x, r, \theta)]\partial_r\bar{u} - \frac{1}{r^2}L_\theta(x, r)\bar{u}.$$

Lemma:

The operator $L_\theta(x, r)$ is a Laplacian on \mathbb{S}_{n-1} for a particular metric which depends on r .

Proof on the Lemma:

We have $\Delta_{\mathbb{S}_r(x)} = \Delta_{i_{x,r}^*(g)}$ with $i_{x,r}$ the identity map from $\mathbb{S}_r(x)$ in M and $i_{x,r}^*(g) = \tilde{g}$ the induced metric on the submanifold $\mathbb{S}_r(x)$ of M .

The map \exp_x induce a diffeomorphism from $\mathbb{S}_r(x)$ into \mathbb{S}_{n-1}^r , we have:

$$\Delta_{\mathbb{S}_r(x)}u|_{\mathbb{S}_r(x)} = \Delta_{i_{x,r}^*(g)}u|_{\mathbb{S}_r(x)} = \Delta_{\tilde{g}}u|_{\mathbb{S}_r(x)} = \Delta_{\exp_x^*(\tilde{g}), \mathbb{S}_{n-1}^r}u\circ\exp_x(v),$$

Let us consider \tilde{z} the map from \mathbb{S}_{n-1} into \mathbb{S}_{n-1}^r defined by $\theta \rightarrow r\theta$. Thus,

$$\Delta_{\exp_x^*(\tilde{g})}u\circ\exp_x(v) = \Delta_{\tilde{z}^*[\exp_x^*(\tilde{g})]}u\circ\exp_x(r\theta),$$

For a chart (ψ, U^k) on \mathbb{S}_{n-1} , the polar chart in x is, $(\varphi_0, [0, \epsilon_0] \times U^k)$, where $\varphi_0 = \exp_x \circ \tilde{z} \circ \psi^{-1}$. The g_{jl}^k are, by definition:

$$g_{jl}^k(r, \theta) = g_{\exp_x(r\theta)}(\partial_{j, \varphi_0}, \partial_{l, \varphi_0}),$$

with, $\partial_{j, \varphi_0}, \partial_{l, \varphi_0}$ the canonical vector fields for the chart $(\varphi_0, [0, \epsilon_0] \times U^k)$.

By definition, take h a function defined in a neighborhood of x :

$$\partial_{j, \varphi_0}(h) = \frac{\partial(h \circ \varphi_0)}{\partial \bar{\theta}_j} = \frac{\partial(h \circ \exp_x \circ \tilde{z} \circ \psi^{-1})}{\partial \bar{\theta}_j},$$

with $\psi(\theta) = (\bar{\theta}_1, \dots, \bar{\theta}_{n-1})$ and $\theta_0 = r$ (angular and radial derivations).

We look to the angular components, ∂_{j,φ_0} with $1 \leq j \leq n-1, j \in \mathbb{N}$, thus,

$$\partial_{j,\varphi_0}(h) = \frac{\partial(hoi_{x,r}o\exp_x o\tilde{z}o\psi^{-1})}{\partial\bar{\theta}_j},$$

$$\frac{\partial(ho\varphi_0)}{\partial\bar{\theta}_j} = \frac{\partial(hoi_{x,r}o\varphi_0)}{\partial\theta_j},$$

Thus, if write $\bar{\partial}_j, \bar{\partial}_l$ as a canonical vector fields of the unit sphre in the chart (ψ, U^k) , we obtain, we use d for the differential,

$$\partial_{j,\varphi_0} = d(i_{x,r}o\exp_x o\tilde{z})(\bar{\partial}_j),$$

Thus,

$$g_{jl}^k(r, \theta) = g_{\exp_x(r\theta)}[d(i_{x,r}o\exp_x o\tilde{z})(\bar{\partial}_j), d(i_{x,r}o\exp_x o\tilde{z})(\bar{\partial}_l)],$$

we use the definition of the pull-back:

$$g_{jl}^k(r, \theta) = \tilde{z}^*[\exp_x^*[i_{x,r}^*(g)]]_{\theta}(\bar{\partial}_j, \bar{\partial}_l) = \tilde{z}^*[\exp_x^*(\tilde{g})](\bar{\partial}_j, \bar{\partial}_l),$$

Thus we have the component j, l of the mteric $\tilde{z}^*[\exp_x^*(\tilde{g})]$ on the unit sphere \mathbb{S}_{n-1} .

Finaly we have locally and globally:

$$-\Delta_{\tilde{z}^*[\exp_x^*(\tilde{g})], \mathbb{S}_{n-1}} u o \exp_x(r\theta) = \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} [\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j} u o \exp_x(r\psi^{-1})],$$

Now, if we consider the metric on \mathbb{S}_{n-1} defined by,

$$g_{x,r,\mathbb{S}_{n-1}} = r^{-2} \tilde{z}^*[\exp_x^*(\tilde{g})],$$

this metric is well-defined and is such that:

$$g_{x,r,\mathbb{S}_{n-1}}{}_{jl} = r^{-2} g_{jl}^k = \tilde{g}_{jl}^k,$$

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